Existence and Uniqueness of the Solution for Fractional Sturm - Liouville Boundary Value Problem

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Abstract:

In this paper, we prove the existence and uniqueness of the solution for a fractional Sturm-Liouville boundary value problem. We give two results, one based on Banach fixed point theorem and the other based on Schaefer's fixed point theorem.

وجود و وحدانية الحل لمسألة شتورم – ليوفيل الحدودية الكسرية

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ملخص البحث:

في هذا البحث سوف ندرس وجود و وحدانية الحل لمعادلة تفاضلية كسرية من نوع شتورم– ليوفيل ذات رتبة كسرية مع شروط حدودية ، حيث سنعطي نتيجتين:الأولى حسبمبرهنة بناخ للنقطة الثابتة والأخرى حسبمبرهنة شافير للنقطة الثابتة.

1- Introduction

Consider the following fractional boundary value problem

$$D^{\alpha}\big(p(t)y'(t)\big) + q(t)y(t) + f\big(t, y(t)\big) = 0 \quad (1)$$

$$a\, y(0) - b\, y'(0) = 0$$

$$c\, y(T) + d\, y'(T) = 0 \quad (2)$$

Where ${}^{C}D^{\alpha}$is the standard Caputo derivative, and $0 < \alpha < 1$and $t \in J = [0, T]$, $y \in C(J, R)$The Banach space with norm: $\|y\|_{\infty} = sup\{|y(t)|: t \in J\}$ and the functions $p: J \to R, q: J \to R, f: J \times R \to R$are continuous functions,$p(t) > 0$ for all $t \in J$and a, b, c, d are constants.

The problem of the existence and uniqueness of the solution for fractional differential equations have been considered by many authors; see for example [1], [2], [3], [٦], [٧] ,[9],[12].The existence and uniqueness problems of fractional nonlinear differential equations as a basic theoretical part of some applications are investigated also by many authors (see for examples [2], [11], and [12]). It arises in many fields like electronic, fluid dynamics, biological models, and chemical kinetics. A well-known example is the equations of basic electric circuit analysis. Some results for fractional differential inclusions can be found in the book by Plotnikov [10].

Very recently some basic theory for the initial value problems of fractional differential
Equations involving Riemann-Liouville differential operator has been discussedby Lakshmikantham and Vatsala [13, 14and 15].

In [8] the authors studied the existence of solutions for first order boundary value problems (BVP for short), for fractional order differential equations: $D^{\alpha}y(t) = f(t, y(t))$for each $t \in J = [0, T]$, 0 <α<1, with boundary condition $a\,y(0) + b\,y(T) = c$ by using Banach fixed point theorem and Schaefer's fixed point theorem .

Sturm-Liouville problem $(py')' + qy + g(y) = 0$with periodic nonlinearities was studied in [11], and in [2] the author studied the third-order Sturm-Liouville boundaryvalue problem, with p-Laplacian, $\left(\varphi_p(y'')\right)' + f(t, y) = 0, \alpha y(0) - \beta y'(0) = 0,\ \gamma y(1) + \delta y'(1) = 0, y''(0) = 0$

In this paper, we present existence results for the fractional Sturm-Liouville problem (1)-(2). In Section 3, we give two results, one based on Banach fixed point theorem (Theorem 3.1) and theother based on Schaefer's fixed point theorem (Theorem 3.2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from fractional calculus theories which are used throughout this paper. These definitions can be found in the recent literature.

Definition 2.1.[4]Let $\alpha > 0$, for a function$y : (0, +\infty) \to R$. the fractional integral of order α of y is defined by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}y(s)ds$$

Provided the integral exists.

Definition 2.2.The Caputo derivative of a function $y: (0, +\infty) \to R$ is given by

$$^{c}D^{\alpha}y(t) = I^{n-\alpha}\left(D^{n}y(t)\right) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t - s)^{n-\alpha-1}y^{(n)}(s)ds$$

Provided the right side is point wise defined on$(0, +\infty)$,where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real numberα.

The properties of the above operators can be found in [5] and the general theoryoffractional differential equations can be found in [4].Γdenotes the Gamma function:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t}\, t^{\alpha-1}\ dt$$

The Gamma function satisfies the following basic properties:

(1) For any $n \in R$

$\Gamma(n+1) = n\Gamma(n)$and if $n \in Z$ $then$ $\Gamma(n) = (n-1)!$

(2)For any$1 < \alpha \in R$, then

$$\frac{\alpha+1}{\Gamma(\alpha+1)} = \frac{\alpha+1}{\alpha\,\Gamma(\alpha)} < \frac{2}{\Gamma(\alpha)}$$

From Definition (2.2) we can obtain the following lemma.

Lemma 2.3.Let $0 < n-1 < \alpha < n$. If we assume $y \in C^n(0,T)$, the fractional differential equation ${}^{c}D^{\alpha}y(t) = 0$has a unique solution

$$y(t) = y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \cdots + \frac{y^{(n)}(0)}{n!}t^n$$

Where$n = [\alpha] + 1$

***Theorem 2.4. (Schaefer's Theorem)*[18]**.Let X be a Banach space and let $T : X \to X$ be a completely continuous operator Then either

(a) T has a fixed point, or

(b) the set $\varepsilon = \{x \in X | x = \lambda Tx, \lambda \in (0,1)\}$ is unbounded

Theorem 2.5. (Arzela-Ascoli Theorem).[17]For $A \in C[0,1]$, A is compact if and only if A isclosed, bounded, and equicontinuous.

Compact operators on a Banach space are always completely continuous. [16]

Theorem 2.6. (Banach's Fixed Point Theorem).[17]Let K be Banach space, and let $F : K \to K$ be a contraction mapping, Then F has a unique fixed point, i.e. there exists a unique $A \in K$ such that $F(A) = A$

Lemma2.7.Let $0 < \alpha < 1$and let $p{:}J \to R, q{:}J \to R, h : J \to R$are continuousfunctions, $p(t) > 0$ for all $t \in J$and a, b, c, d are constants. A function$y$is a solution of the fractional Sturm-leoville problem

$$\begin{cases} D^{\alpha}\big(p(t)y'(t)\big) + q(t)y(t) + h(t) = 0 \\ a\,y(0) - b\,y'(0) = 0 \\ c\,y(T) + d\,y'(T) = 0 \end{cases}$$

If and only if$y$is a solution of the following fractional integral equation:

$$y(t) = y(0)[1 + \frac{a}{b}\int_0^t \frac{p(0)}{p(s)}ds\,] - \int_0^t \frac{1}{p(s)}[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\big(q(r)y(r) + h(r)\big)dr\Big]ds \qquad (3)$$

Where

$$y(0) = \frac{c\int_0^T \frac{1}{p(s)}[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left(q(r)y(r)+h(r)\right)dr]ds}{C+\frac{a}{b}\left[c\int_0^T \frac{p(0)}{p(s)}ds + d\frac{p(0)}{p(T)}\right]} + \frac{d\frac{1}{p(T)}[\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left(q(s)y(s)+h(s)\right)ds]}{C+\frac{a}{b}\left[c\int_0^T \frac{p(0)}{p(s)}ds + d\frac{p(0)}{p(T)}\right]}$$

Proof.Assume y Satisfied (1) and (2) then by lemma (2. 3)

$$p(t)y'(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(q(s)y(s)+h(s)\right)ds] = c$$

$$y'(t) = \frac{c}{p(t)} - \frac{1}{p(t)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(q(s)y(s)+h(s)\right)ds$$

when $t = 0$ we get $y'(0) = \frac{c}{p(0)} \Rightarrow c = y'(0)\,p(0)$ then

$$y'(t) = \frac{y'(0)\,p(0)}{p(t)} - \frac{1}{p(t)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(q(s)y(s) + h(s)\right)ds \qquad (4)$$

By Integrating we get

$$y(t) = y(0) + y'(0)\int_0^t \frac{p(0)}{p(s)}ds - \int_0^t \left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left(q(r)y(r)+h(r)\right)dr\right)ds$$

By condition $a\,y(0) - b\,y'(0) = 0 \Rightarrow y'(0) = \frac{a}{b}y(0)$ then

$$y'(t) = y(0)\frac{a}{b}\frac{p(0)}{p(t)} - \frac{1}{p(t)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(q(s)y(s)+h(s)\right)ds$$

and

$$y(t) = y(0) + y(0)\frac{a}{b}\int_0^t \frac{p(0)}{p(s)}ds - \int_0^t \left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left(q(r)y(r)+h(r)\right)dr\right)ds$$

$$y(t) = y(0)\left(1 + \frac{a}{b}\int_0^t \frac{p(0)}{p(s)}ds\right) - \int_0^t \left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\big(q(r)y(r) + h(r)\big)dr\right)ds \quad (5)$$

By the condition $c\,y(T) + d\,y'(T) = 0$ then

$$c\left[y(0) + y(0)\frac{a}{b}\int_0^T \frac{p(0)}{p(s)}ds - \int_0^T \left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\big(q(r)y(r) + h(r)\big)dr\right)ds\right] + d\left[y(0)\frac{a}{b}\frac{p(0)}{p(T)} - \frac{1}{p(T)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\big(q(s)y(s) + h(s)\big)ds\right] = 0$$

$$y(0) = \frac{c\int_0^T \left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\big(q(r)y(r) + h(r)\big)dr\right)ds}{c + \frac{a}{b}\left(c\int_0^T \frac{p(0)}{p(s)}ds + d\frac{p(0)}{p(T)}\right)} + \frac{\frac{d}{p(T)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\big(q(s)y(s) + h(s)\big)ds}{c + \frac{a}{b}\left(c\int_0^T \frac{p(0)}{p(s)}ds + d\frac{p(0)}{p(T)}\right)}$$

The converse obtainedby substituting (5) in (1)-(2).

3. Main Result

In this section, we give the existence and uniqueness of the solutions for problem (1)-(2).

Our first result based on Banach fixed point theorem.

Theorem 3.1Assume that:

(H_1) There existsa positive constant $K > 0$such that

$$|f(t,u) - f(t,v)| \le K|u - v|$$

For each $t \in J$ and all $u, v \in R$

(H_2)There exists a positive constant Qsuch that

$$q(t) \le Q$$

For all$t \in J$

If $\theta = (Q + K)\int_0^T \frac{1}{p(s)}\frac{s^\alpha}{\Gamma(\alpha + 1)}\, ds < 1$ (6)

then (1)-(2) has a unique solution on J.

Proof. we transform the problem (1)-(2) into fixed point problem .

Conseder the operter F: $C(J, R) \to C(J, R)$defined by:

$$F(y)(t) = y(0)[1 + \frac{a}{b}\int_0^t \frac{p(0)}{p(s)} ds] - \int_0^t \frac{1}{p(s)}[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\Big(q\big(r\big)y(r) + f\big(r, y(r)\big)\Big) dr\Big] ds \qquad (7)$$

Cearly, any fixed point of the operater F is asolution of the problem (1)-(2).

We shall use the Banach contraction principle to prove that F has afixed point.

Let $x, y \in C(J, R)$,Then for each $t \in J$ we have

$$|Fy(t) - Fx(t)| \leq \int_0^t \frac{1}{p(s)}[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\ [|q(r)||y(r) - x(r)| + \big|f\big(r, y(r)\big) - f(r, x(r))\big| dr]ds$$

$$\leq \int_0^t \frac{1}{p(s)}[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\ [Q|y(r) - x(r)| + K|y(r) - x(r)|dr]ds$$

$$\leq \int_0^t \frac{1}{p(s)}(Q + K)\|y - x\|_\infty[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\ dr]ds$$

$$\leq (Q + K)\int_0^t \frac{1}{p(s)}\frac{s^\alpha}{\Gamma(\alpha + 1)}\ ds\ \|y - x\|_\infty$$

$$= \theta\|y - x\|_\infty$$

Therfore

$$\|F(y) - F(x)\|_\infty \leq \theta\|y - x\|_\infty$$

Consequently by (6) ,F is acontraction . As consequnce of Banach fixed point theorem, we deduce that F has a unique fixed point .which is the solution of the problem(1)-(2).
Our second result basedon the Schaefer's fixed point theorem
Theorem 3.2Assume that
(H_3) The funcation $f: J \times R \to R$ is continuous.
(H_4) There existsa positive constant $M > 0$, $N > 0$such that

$$\|f(t,u)\| \leq M$$

For each $t \in J$ *and* $u \in R$,and

$$\int_0^T \frac{1}{p(s)}\, ds \leq N$$

Thenthe problem(1)-(2)has at least one uniqe solution on J.
Proof. We shall use Schaefer's fixed point theorem to prove that F defined by (7)has a fixed point.
The proof will be given in several steps.
Step 1: F is continuous.
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,R)$. Then for each $t \in J$

$$|F(y_n)(t) - F(y)(t)|$$

$$\leq \int_0^t \frac{1}{p(s)}\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left[|q(r)||y_n(r) - y(r)| + \left|f(r,y_n(r)) - f(r,y(r))\right|\right] dr\right) ds$$

$$\leq \int_0^t \frac{1}{p(s)}\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left[Q|y_n(r) - y(r)| + \sup_{r\in J}\left|f(r,y_n(r)) - f(r,y(r))\right|\right] dr\right) ds$$

$$\leq \left(Q\|y_n - y\|_\infty + \left\|f(.,y_n(.)) - f(.,y(.))\right\|_\infty\right)\int_0^t \frac{1}{p(s)}\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\, dr\right) ds$$

$$\leq \left(Q\|y_n - y\|_\infty + \left\|f(.,y_n(.)) - f(.,y(.))\right\|_\infty\right)\int_0^t \frac{1}{p(s)}\left(\frac{s^\alpha}{\Gamma(\alpha+1)}\right) ds$$

$$\leq \left(Q\|y_n - y\|_\infty + \left\|f\left(.,y_n(.)\right) - f(.,y(.))\right\|_\infty\right)\frac{T^\alpha}{\Gamma(\alpha+1)}\int_0^T \frac{1}{p(s)}ds$$

$$\leq \left(Q\|y_n - y\|_\infty + \left\|f\left(.,y_n(.)\right) - f(.,y(.))\right\|_\infty\right)\frac{T^\alpha}{\Gamma(\alpha+1)}N$$

Since f is a continuous function and $y \in C(J,R)$, $y_n \to y$, we have

$$\|F(y_n) - F(y)\|_\infty \leq \left(Q\|y_n - y\|_\infty + \left\|f\left(.,y_n(.)\right) - f\left(.,y(.)\right)\right\|_\infty\right)\frac{T^\alpha}{\Gamma(\alpha+1)}N \to 0 \; As n \to \infty$$

Step 2. F maps bounded sets into bounded sets in $C(J,R)$.
Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant l
such that for each $y \in B_\eta = \{ y \in C(J,R) : \|y\|_\infty < \eta\}$; we have $\|F(y)\|_\infty < l$.
By (**H$_3$**) we have for each $t \in J$

$$|Fy(t)| \leq y(0)\left(1 + \left|\frac{a}{b}\right|\int_0^t \frac{p(0)}{p(s)}ds\right) + \int_0^t \frac{1}{p(s)}\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left[|q(r)||y(r)| + \left|f\left(r,y(r)\right)\right|\right]dr\right)ds$$

$$\leq y(0)\left(1 + \left|\frac{a}{b}\right|\int_0^t \frac{p(0)}{p(s)}ds\right) + \int_0^t \frac{1}{p(s)}\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}[Q\eta + M]\,dr\right)ds$$

$$\leq y(0)\left(1 + \left|\frac{a}{b}\right|\int_0^t \frac{p(0)}{p(s)}ds\right) + (Q\eta + M)\int_0^t \frac{1}{p(s)}\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\,dr\right)ds$$

$$= y(0)\left(1 + \left|\frac{a}{b}\right|\int_0^t \frac{p(0)}{p(s)}ds\right) + (Q\eta + M)\int_0^t \frac{1}{p(s)}\left(\frac{s^\alpha}{\Gamma(\alpha+1)}\right)ds$$

$$\leq y(0)\left(1+\left|\frac{a}{b}\right|\int_0^t \frac{p(0)}{p(s)}ds\right)+(Q\eta+M)\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)\int_0^T \frac{1}{p(s)}ds$$

$$\leq y(0)\left(1+\left|\frac{a}{b}\right|p(0)N\right)+(Q\eta+M)\frac{T^\alpha}{\Gamma(\alpha+1)}N$$ Thus

$$\|F(y)\|_\infty \leq y(0)\left(1+\left|\frac{a}{b}\right|p(0)N\right)+(Q\eta+M)\frac{T^\alpha}{\Gamma(\alpha+1)}N := l$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J,R)$.
Let $t_1, t_2 \in J$, $t_1 < t_2$. B_η be a bounded set of $C(J,R)$ as in Step 2, and let $y \in B_\eta$,
then
$|Fy(t_1) - Fy(t_2)|$

$$= \left| y(0)\frac{a}{b}\int_0^{t_1}\frac{p(0)}{p(s)}ds - \int_0^{t_1}\left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left(q(r)y(r) + f(r,y(r))\right)dr\right)ds - y(0)\frac{a}{b}\int_0^{t_2}\frac{p(0)}{p(s)}ds + \int_0^{t_2}\left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left(q(r)y(r) + f(r,y(r))\right)dr\right)ds \right|$$

$$\leq \left|y(0)\frac{a}{b}\right|\left(\int_{t_1}^{t_2}\frac{p(0)}{p(s)}ds\right) + \int_{t_1}^{t_2}\left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left[|q(r)||y(r)| + |f(r,y(r))|\right]dr\right)ds$$

$$\leq \left|y(0)\frac{a}{b}\right|\left(\int_{t_1}^{t_2}\frac{p(0)}{p(s)}ds\right) + \int_{t_1}^{t_2}\left(\frac{1}{p(s)\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}[Q\eta+M]dr\right)ds$$

$$= \left|y(0)\frac{a}{b}\right|\left(\int_{t_1}^{t_2}\frac{p(0)}{p(s)}ds\right) + (Q\eta + M)\int_{t_1}^{t_2}\left(\frac{1}{p(s)}\frac{s^{\alpha}}{\Gamma(\alpha+1)}\right)ds$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.
As a consequenceof Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that$F: C(J,R) \to C(J,R)$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$A = \{y \in C(J;R): y = \lambda F(y) \quad for\ some\ 0 < \lambda < 1\}$$

is bounded.

Let $y \in A$, then $y = \lambda F(y)$ for some $0 < \lambda < 1$.Thus, for each $t \in J$ we have

$$y(t) = \lambda \left| y(0)[1 + \frac{a}{b}\int_0^t \frac{p(0)}{p(s)}ds] - \int_0^t \frac{1}{p(s)}[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left(q\left(r\right)y(r) + f\big(r,y(r)\big)\right)dr\Big]ds \right|$$

This implies by (**H**$_3$) that for each$t \in J$ we have

$$|Fy(t)| \leq |y(0)|\left[1 + \left|\frac{a}{b}\right|\int_0^t \frac{p(0)}{p(s)}ds\right] + \int_0^t \frac{1}{p(s)}\left[\frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}\left(|q(r)||y(r)| + \left|f\big(r,y(r)\big)\right|\right)dr\right]ds$$

$$\leq |y(0)|\left[1 + \left|\frac{a}{b}\right|p(0)N\right] + (Q\eta + M)\frac{T^{\alpha}}{\Gamma(\alpha+1)}N$$

Thus for every $t \in J$, we have

$$\|Fy(t)\|_{\infty} \leq |y(0)|\left[1 + \left|\frac{a}{b}\right|p(0)N\right] + (Q\eta + M)\frac{T^{\alpha}}{\Gamma(\alpha+1)}N := \ell$$

This shows that the set A is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that *F*has a fixed point which is a solution of the problem (1) - (2).

4. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional BVP

$$D^{\alpha}\left((2-t^2)y'(t)\right) + sin(2\pi t)y(t) + \frac{|y(t)|}{|y(t)|+1}$$
$$= 0 \qquad (8)$$
$$y(0) - y'(0) = 0$$
$$y(1) + y'(1) = 0 \qquad (9)$$

Here, $p(t) = 2 - t^2$, $q(t) = \sin(2\pi t)$, $f(t,y) = \frac{|y|}{|y|+1}$ for all $t \in [0,1]$,

and $a = b = c = d = 1$

Then we have:

$$|q(t)| = |\sin(2\pi t)| \leq 1 := Q$$
$$\left|\frac{\partial f(t,y)}{\partial y}\right| = \frac{1}{y^2+1} \leq 1 := K$$
$$|f(t,y_1) - f(t,y_2)| \leq |y_1 - y_2|$$
$$\therefore \theta = (Q+K)\int_0^1 \frac{1}{p(s)}\frac{s^{\alpha}}{\Gamma(\alpha+1)}\, ds = (1+1)\int_0^1 \frac{1}{2-s^2}\frac{s^{\alpha}}{\Gamma(\alpha+1)}\, ds$$
$$\leq 2\int_0^1 \frac{1}{2}\frac{s^{\alpha}}{\Gamma(\alpha+1)}\, ds = \frac{1}{\Gamma(\alpha+2)} < 1$$

Then (H_1) and (H_2) are satisfied with $Q = 1\ and\ \theta = \frac{1}{\Gamma(\alpha+2)} < 1$.

Then by Theorem 3.1the fractional BVP (8)-(9) has a unique solution on[0, 1].

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