On Generalized π - Regular Rings

Assistant Lecturer
Najlaa T.Jassim
College of Basic Education - University of Mosul

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Abstract:

The purpose of this paper is to study right generalized π-regular rings and give some of it is properties. Also, we proved:

1- Let R be a right generalized π-regular ring without zero divisors element. Then R is a division ring.

2- 2- Let R be a ring with $l(a^n) \subseteq r(a^n)$, for every $a \in R$ and $n \in z^+$. Then R is regular. If and only if R is a right generalized π-regular ring.

حول الحلقات المنتظمة المعممة من النمط π

م.م. نجلاء طارق جاسم
كلية التربية الأساسية -جامعة الموصل

ملخص البحث :

الغرض من هذا البحث هو دراسة الحلقات المنتظمة المعممة من النمط π وإعطاء بعض من هذه الخواص. كذلك برهنا:

١. لتكن R حلقة منتظمة معممة من النمط π يمنى ولا تحتوي على عنصر قاسم للصفر عند ذلك تكون R حلقة القسمة.

٢. لتكن R حلقة و $l(a^n) \subseteq r(a^n)$, لكل $a \in R$ و $n \in z^+$ عندها تكون R حلقة منتظمة اذا وفقط اذا كانت R حلقة منتظمة معممة من النمط π يمنى.

1. Introduction:

Throughout this paper, R denote an associative rings with identity. The right singular ideal and the Jacobson radical of a ring R are denoted by Y(R) and J(R), respectively. We say that R is right duo ring if all right ideal are ideal of R[6]. We say that ring R is regular if for all $a \in R$, there exists $b \in R$ such that a=aba. This concept was first introduced by von Neumann[8] and Chen [4]. As a generalization of this concept Azumaya[3] introduced π-regular rings as a ring R which π-regular if for every $a \in R$, $\exists n \in z^+$ and $b \in R$ such that $a^n = a^n b a^n$ An ideal I of the ring R is said to be regular if for all $a \in I$, there exists $b \in I$ such that a=aba.

2. Generalized π-regular ring:

Definition 2.1: [4]

Let $0 \neq a \in R$, we say that a is a right (left)generalized π-regular, if there exists a positive integer n such that $a^n = aba^n$ ($a^n ba$), for some,$b \in R$.

We call to the ring R is said to be a right (left) generalized π-regular if and only if every element in R is a right (left) generalized π-regular. If R is right and left generalized π-regular,we called R to be generalized π-regular ring.

Every regular and π-regular is generalized π-regular.

Examples:

1- Z_6, Z_{10}, Z_{14} and Z_{15} is a generalized π-regular ring.

2- Let Z_2 be a ring of integers modulo 2, and let $R = \begin{bmatrix} z_2 & z_2 \\ 0 & z_2 \end{bmatrix}$ .Therefore R is a generalized π-regular ring.

Definition 2.2:

A right ideal I of a ring R is said to be right (left) generalized π-regular ideal if and only if for all $b \in I$, b is right (left) generalized π-regular element.

Proposition 2.3 :

Let R be a right generalized π-regular ring, then every two sided ideal I of R is a right generalized π-regular.

Proof :

Let I be an ideal of ring R. For any $x \in I$, since R is right generalized π-regular,there exists a positive integer n and $y \in R$ such that $x^n = xyx^n$, $x^n = xyxyx^n$, we set z=yxy, then we have $x^n = xzx^n$ with $z \in I$ (since $x \in I$ and I is an ideal). Hence I is a right generalized π-regular .

Proposition 2.4:

Let R be a ring if every principal right ideal is a right generalized π-regular, then R is a right generalized π-regular.

Proof :

It is clear.

Theorem 2.5:

Let R be a right generalized π-regular ring and I is a right ideal. Then R/I is a right generalized π-regular .

Proof :

Let $a + I \in R/I$, where $a \in R$. Since R is a generalized π-regular ring, then there exists a positive integer n such that $a^n = aba^n$ for some $b \in R$, then $a^n + I = aba^n + I = (a + I)(b + I)(a^n + I)$.Therefore, R/I is a right generalized π-regular ring .

Theorem 2.6:

Let I be an ideal of a ring R. If R/I is a right generalized π-regular ring and I is regular, then R is a right generalized π-regular ring.

Proof :

Let $a \in R$, then $a + I \in R/I$, if $a + I = I$ we have $a \in I$, since I is a regular, there exists $d \in I$ such that $a=ada$, for any positive integer m, $a^m = ada^m$ and done for all $a \in I$, now if $a + I \neq I$, so there exists a positive integer n $(1 \neq n)$ such that

$(a + I)^n = (a + I)(b + I)(a + I)^n$ for some $b \in R$, $a^n + I = aba^n + I$. Therefore, $a^n - aba^n \in I$, since I is regular, then there exist $c \in I$ such that

$a^n - aba^n = (a^n - aba^n)c(a^n - aba^n)$

$= (a^n c - aba^n c)(a^n - aba^n)$

$= a^n c a^n - a^n c ab a^n - a b a^n c a^n + a b a^n c a b a^n$

$a^n = a^n c a^n - a^n c a b a^n - aba^nca^n + a b a^n c a b a^n + a b a^n$

$a^n = a(a^{n-1}c - a^{n-1}cab - ba^n c + ba^n cab + b) a^n$

set $h = a^{n-1}c - a^{n-1}cab - ba^n c + ba^n cab + b \in R$

So $a^n = a h a^n$. Therefore, R is a right generalized π-regular ring .

Proposition 2.7:

Let R be a right generalized π-regular ring. Then :

(1) J(R) is nil ideal.

(2) Y(R) is nil ideal.

Proof (1) :

Let $a \in J(R)$. Since R is a right generalized π-regular ring, then there exists, a positive integer n such that $a^n = a ba^n$ for some $b \in R$, $(1-ab)a^n = 0$, Since $a \in J(R)$ then $1-ab$ is invertible. Therefore must $a^n = 0$ for all $a \in J(R)$. So J(R) is nil ideal.

Proof (2):

Let $a\in Y(R)$. Since R is a right generalized π-regular ring ,then there exists a positive integer n such that $a^n=aba^n$ for some $b\in R$. Since $a\in Y(R)$ then r(ab) is essential right ideal of R. So $r(ab)\cap a^nR\neq 0$.There exists $0\neq x\in r(ab)\cap a^nR$, abx=0 and x= $a^nr=aba^nr=abx=0$, but r(ab) is essential.Therefor $a^nR=0$ which lead us to $a^n=0$ for all $a\in Y(R)$. So Y(R) is a nil ideal.

A ring R is called reversible if ab=0 implies ba=0 for every $a, b\in R$ [5].

Theorem 2.8:

Let R be a reversible ring and $a\in R$. If r(a) is right generalized π-regular ring then for any positive integer m >1 and $x\in r(a^m)$, $a^{m-1}x$ is a nilpotent element.

Proof :

Let $x\in r(a^m)$. Then $a^m x=0$, $aa^{m-1}x=0$, and hence $a^{m-1}x\in r(a)$. Since r(a) is generalized π-regular, there exists a positive integer n with $(1\neq n)$ and $y\in r(a)$ such that $(a^{m-1}x)^n = a^{m-1}x\, y(a^{m-1}x)^n$ we have y a=0 (since $y\in r(a)$, ay=0=y a, R is reversible ring) so $(a^{m-1}x)^n=0$, it is a nilpotent element.

3.The connection between generalized π-regular rings and other rings:

Proposition 3.1:

Let R be a reversible ring. Then R is a right generalized π-regular ring if and only if $aR+r(a^n)=R$ for all $a\in R$ and $n\in Z^+$.

Proof :

Let R be a right generalized π- regular, and $a\in R$, then there exists a positive integer n such that $a^n=aba^n$, for some $b\in R$. If $aR+r(a^n)\neq R$, then there exists a maximal right ideal M of R such that $aR+r(a^n)\subseteq M$. Since

$a^n=aba^n$, $(1-ab)a^n=0$, since R is reversible $a^n(1-ab)=0$, $1-ab\in r(a^n)\subseteq M$. So $1\in M$, which is a contradiction therefore $aR+r(a^n)=R$ for all $a\in R$.

Conversely : Assume that $aR+r(a^n)=R$, for all $a\in R$ and $n\in z^+$. Hence $ab+d=1$, $a\in R$ and $d\in r(a^n)$ so $aba^n+da^n=a^n$, since $d\in r(a^n)$, $a^nd=0$, R is reversible $da^n=0$, so $aba^n=a^n$ for all $a\in R$. Therefore R is a right generalized π-regular ring.

Theorem 3.2:

Let R be a right generalized π-regular ring without zero divisors elements, then R is a division ring.

Proof :

Let $0\neq a\in R$. Since R is a right generalized π-regular ring, then there exists a positive integer n such that $a^n = aba^n$, for some $b\in R$, implies that $(1-ab)a^n = 0$. Since R without zero divisors then either $a^n = 0$ or $1-ab=0$. If $a^n = 0$ then a is a zero divisors which a contradiction. Thus $1-ab = 0$, $1= ab$, so a is a right invertible element. Let $b\in R$ and R is a right generalized π-regular ring, then there exists a positive integer $m(m\neq 1)$ such that $b^m = bcb^m$, for some $c\in R$, gives $1=bc$. Now $a=a.1=abc=1c$, so $a=c$, therefore R is a division ring.

Theorem 3.3:

Let R be a ring, $l(a^n)=l(a)$ for all $a\in R$ and any positive integer n. Then the following condition are equivalent.

1- R is regular.

2- R is right generalized π-regular.

Proof :

$1\rightarrow 2$ It is clear.

$2\rightarrow 1$ Let $a\in R$, since R is a right generalized π-regular ring, then there exists a positive integer n such that $a^n=aba^n$, for some $b\in R$, (1-

$ab)a^n = 0$, $(1- b\ a) \in l(a^n) = l(a)$. So $(1-a\ b)a = 0$. $a=aba$ for all $a \in R$. Therefore, R is a regular ring.

A right ideal I of a ring R is said to be GP-ideal, if for all $a \in I$ there exists $b \in I$
such that $a^n = a^n b$[7].

Theorem 3.4:

Let R be a right duo ring. Then R is right generalized π- regular if and only if every principal right ideal is left GP-ideal.

Proof :

Let R be a right generalized π- regular and $a \in R$. Let $x \in aR$, since R is a right generalized π-regular ring, then there exists a positive integer n such that $x^n = xbx^n$, for some $b \in R$, set $xb \notin aR$, so $d \in aR$, $x^n = dx^n$ for all $x \in aR$ then aR is a left GP- ideal.

conversely

Let every principal right ideal is left GP- ideal and $a \in R$, since $a \in aR$, there exists a positive integer n such that $a^n = ba^n$, for some $b \in aR$, now there exists $c \in R$ such that $b=ac$, therefore $a^n = aca^n$ that is hold for all $a \in R$. Then R is right generalized π- regular ring.

A ring R is said to be right (left) semi π- regular if for all $a \in R$, there exists a positive integer n and an element $b \in R$, such that $a^n = a^n b$ ($a^n = ba^n$) and $r(a^n) = r(b)$ ($l(a^n) = l(b)$)[2].

Theorem 3.5:

Let R be a right generalized π- regular ring and for any non zero element x,y in a ring R, $Rxy \cap l(a^n) = 0$ for any positive integer n. Then R is a left semi- regular ring.

Proof :

Let $a \in R$, since R is a right generalized π-regular, then there exists a positive integer n such that $a^n=aba^n$, for some $b \in R$. Set that c=ab, $x^n=cx^n$, now we to show that $l(a^n)=l(c)$. Let $x \in l(c)$, then xc=0, xab=0, $xaba^n=0$, $xa^n=0$,$x \in l(a^n)$, which yield to $l(c) \subseteq l(a^n)$-----(1). Let $y \in l(a^n)$, $ya^n=0$. (since $a^n=aba^n$), so $yaba^n=0$(c=ab), $yca^n=0$, $yc \in l(a^n)$ also $yab=yc \in Rab$, but $Rab \bigcap l(a^n)=0$, must yc=0, $y \in l(c)$ which yield to $l(a^n) \subseteq l(c)$ -----(2). From (1)and (2) we get that $l(a^n)=l(c)$. Therefore R is a left semi- regular ring.

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