Determination and Testing the Domination Numbers of Tadpole Graph, Book Graph and Stacked Book Graph Using MATLAB

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Abstract:

A set $S \subseteq V$ is said to be dominating set of G if for every $v \in V - S$ there exists a vertex $u \in S$ such that $uv \in E$. The minimum cardinality of vertices among dominating set of G is called the domination number of G denoted by $\gamma(G)$.

We investigate the domination number of Tadpole graph, Book graph and Stacked Book graph. Also we test our theoretical results in computer by introducing a matlab procedure to find the domination number $\gamma(G)$, dominating set S and draw this graph that illustrates the vertices of domination this graphs. It is proved that:

$$\gamma(T_{m,1}) = \left\lceil \frac{m+2}{3} \right\rceil$$

$$\gamma(T_{m,n}) = \left\lceil \frac{m+2}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil$$

$$\gamma(B_m) = 2$$

$$\gamma(B_{m,n}) = n.$$

Keywords: Dominating set, Domination number, Tadpole graph, Book graph, stacked graph.

إيجاد واختبار العدد المهيمن(المسيطر)Domination number للبيانات تادبول (Tadpole graph)، بيان بوك (Book graph) وبيان ستاكيت بوك (Stacked Book graph) باستعمال الماتلاب

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ملخص البحث :

يقال لأية مجموعة جزئية S من مجموعة الرؤوس V في بيان G بأنها مجموعة مهيمنة Dominating set إذا كان لكل رأس v في المجموعة V-S يوجد رأس u في S بحيث uv ضمن

حافات البيان. ويعرف العدد المهيمن Domination number بأنه أصغر مجموعة أساسية مهيمنة Domination.

في هذا البحث سوف ندرس العدد المهيمن Domination number لبيان تادبول (Tadpole graph) ، بيان بوك (Book graph) وبيان ستاكيت بوك (Stacked Book graph). كذلك تأكدنا من النتائج النظرية التي توصلنا إليها بواسطة الحاسبة باستعمال الماتلاب (matlab) لحساب العدد المهيمن $\gamma(G)$ والمجموعة المهيمنة S ورسمنا البيانات التي توضح الرؤوس المهيمنة، حيث توصلنا إلى النتائج التالية:

$$\gamma(T_{m,1}) = \left\lceil \frac{m+2}{3} \right\rceil$$
$$\gamma(T_{m,n}) = \left\lceil \frac{m+2}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil$$
$$\gamma(B_m) = 2$$
$$\gamma(B_{m,n}) = n.$$

Introduction

Throughout this paper graphs, considered here are finite and simple graph. And [2], [5], [6] are followed for terminology and notation.

Let G=(*V*, *E*) be a simple graph, where *V* and *E* denoted respectively, the set of vertices and edges in G. A subset *S* of *V*(G) is said to be dominating a set of G if for every $v \in V - S$ there exists a vertex $u \in S$ such that $uv \in E(G)$. The minimum cardinality of vertices among dominating set of G is called the domination number of G, denoted by $\gamma(G)$,(see [4]).

For any vertex $v \epsilon V$ the open neighborhood of v is the set:

$N(v) = \{u \in V | uv \in E(G)\}$.

and the closed neighborhood

$N[v] = N(v) \cup \{v\}$.

For a set $S \subseteq V$ the open neighborhood is $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is dominating set if $N[S] = V$,

(see [1]).

Equivalently we defined the domination number of a graph G, $\gamma(G)$ to be the order of smallest dominating set of G. A dominating set S with $|S| = \gamma(G)$ is called a minimum dominating set [12].

The earliest ideas of dominating sets it would seen, data back to the origin of the game chess in India over 400 years ago, in which one studies sets of chess pieces which cover or dominate various opposing pieces or various squares of the chess board [6].

Many applications of domination theory in wireless communication networks [7]. Also dominating sets play an important role in much practical application, for example in the context of distributed computing or mobile ad-hoc networks [3].

Definition(1): The *(m,n)*-Tadpole graph, also called dragon graph denoted by $T_{m,n}$ is the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge.

The *(m,1)*-Tadpole graph is sometimes known as the *m*-Pan graph. The particular cases of the *(3,1)*-and *(4,1)*-Tadpole graphs are known as the Paw graph and Banner graph, respectively [9].

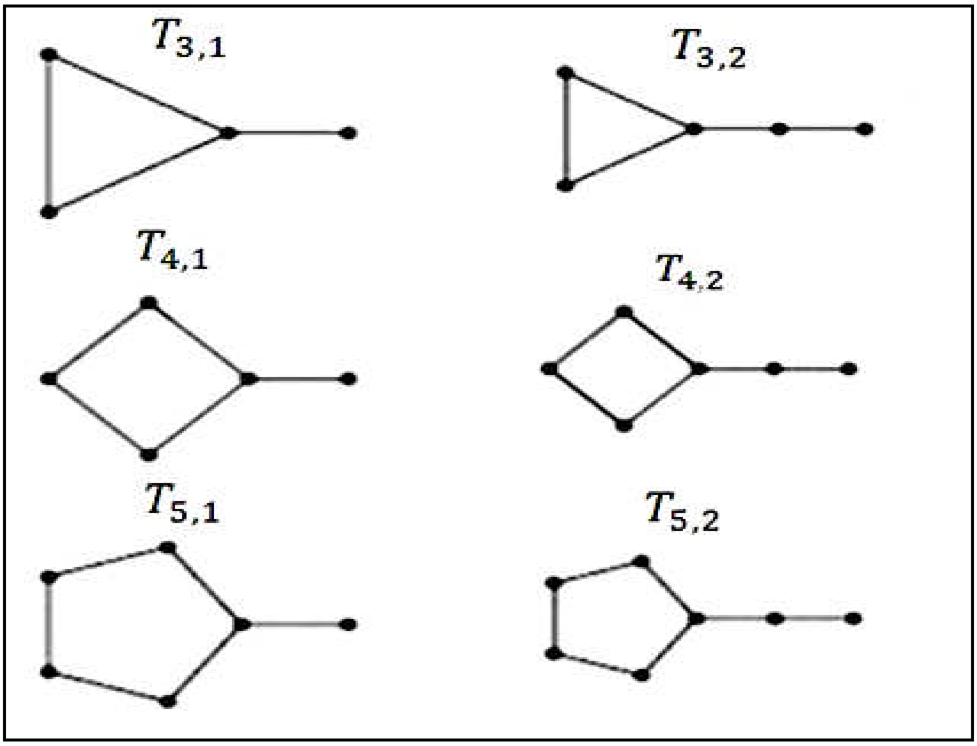

Fig.1: Tadpole graph

Definition (2): The Book graph denoted by B_m is defined as the Cartesian product $S_{m+1} \times P_2$ where S_{m+1} is a star graph and P_2 is the path graph on two vertices [10].

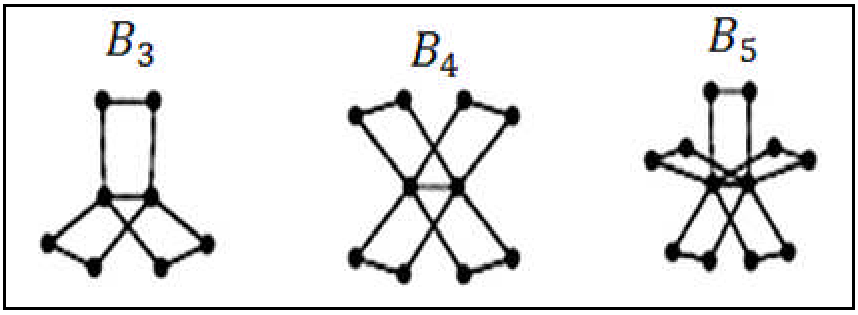

Fig.2: Book graph

Definition (3): The Stacked Book graph denoted by $B_{m,n}$ is defined as the Cartesian product $S_{m+1} \times P_n$ where P_n is the path graph on n vertices [11].

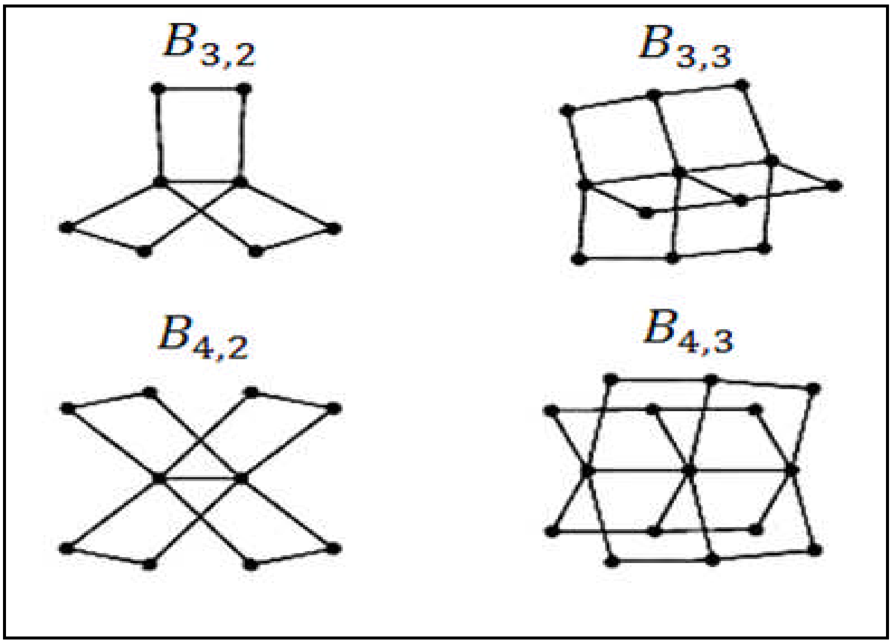

Fig.3: Stacked Book graph

Also we need the following theorem in our work:
Theorem (1): ([8], p.546)
A dominating set D of a graph G is minimal if and only if for each vertex $v \in D$ one of the following conditions satisfied

(i) There exist a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$

(ii) v is an isolated vertex in D .

Note: As usual we use $\lfloor x \rfloor$ for the smallest integer not greater than x.

On domination numbers of Tadpole graph $T_{m,n}$:

In this section we first give the results of the special cases of Tadpole graph $T_{m,n}$ as Paw graph $T_{3,1}$, Banner graph $T_{4,1}$ and Pan graph $T_{m,1}$ then, we generalize the results for any (m) and (n) except the case of Tadpole graph $T_{4,2}$.

<u>Proposition (1):</u> the domination number of Paw graph $T_{3,1}$ is 1 (i.e. $\gamma(T_{3,1}) = 1$).

Proof: Let the vertices of this graph indicated as:
$V(T_{3,1}) = \{v_1, v_2, v_3, v_4\}$. (See Fig.4)

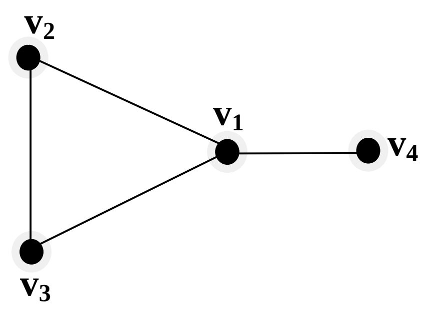

Fig.4

We gave the dominating set of this graph by $S = \{v_1\}$.
The minimality of the set S is very easy since S consist of one vertex that we cannot find a proper subset of S (as theorem (1)).
So S is minimum
$\therefore\ \gamma(T_{3,1}) = |S| = 1$. □

<u>Proposition (2):</u> the domination number of Banner graph $T_{4,1}$ is 2 (i.e. $\gamma(T_{4,1}) = 2$).

Proof: Let the vertices of this graph indicated as:
$V(T_{4,1}) = \{v_1, v_2, v_3, v_4, v_5\}$. (See Fig.5)

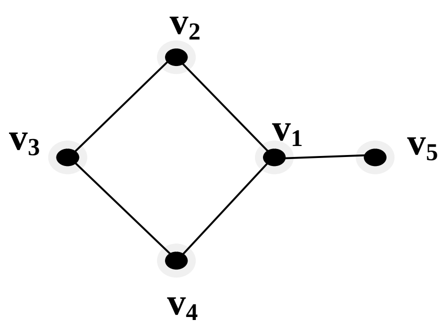

Fig.5

We first give the dominating set of this graph by a set S where: $S = \{v_3, v_5\}$

We prove the minimality of the set S by using the contradiction of theorem (1). So we suppose S not minimal dominating set that is there exist a vertex $v \in S$,such that $S' = S - \{v\}$ is dominating set of $T_{4,1}$.

Now clearly that v either equal $\{v_3\}$ or $\{v_5\}$

So if $v = \{v_3\}$ we have $S' = S - \{v_3\} = \{v_5\}$ from fig.5 clearly there is two vertices $\{v_2, v_4\}$ that adjacent to $\{v_3\}$ not dominating with any vertex in S'

So clearly that S' is not dominating set (by the definition of dominating set).

Similarly, when $v = \{v_5\}$ we have the vertex $\{v_1\}$ that adjacent to $\{v_5\}$ not dominating with any vertex in S'.

So S' is not dominating set

Hence S is minimum

So $\gamma(T_{4,1}) = |S| = 2$. □

Proposition (3): the domination number $T_{4,2}$ is 2

(i.e. $\gamma(T_{4,2}) = 2$).

Proof: is similarly in Proposition (2). □

In proposition (1) and proposition (2) we proof the cases when $m=3$ and $m=4$ as special case, now we generalize for all $m \geq 3$.

Theorem (2): For $m \geq 3$ the domination number of Pan graph $T_{m,1}$ is $\left\lceil \frac{m+2}{3} \right\rceil$

$$\text{(i.e. } \gamma(T_{m,1}) = \left\lceil \frac{m+2}{3} \right\rceil \text{).}$$

Proof: Let the vertices of this graph indicated as:

$V(T_{m,1}) = \{v_1, v_2, v_3, \ldots, v_m, v_{m+1}\}$. (See Fig.6)

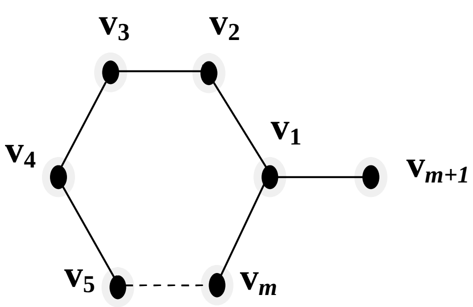

Fig. 6: Pan graph $T_{m,1}$

First we give the dominating set of this graph by a set S where:

$S = \left\{v_{1+3k} \middle| k = 0,1,2,3,\dots,\left\lfloor\frac{m+2}{3}\right\rfloor - 1\right\}$

The minimality of the set S follows from theorem (1) by using the contrary of this theorem.

Assume that S is not minimum dominating set then $\exists v \in S$,such that $S' = S - \{v\}$ is dominating set of $T_{m,1}$.

Therefore $\forall u \in N[v]$,$\exists k \in S - \{v\},\ k \in N[v]$.

For any $v \in S$ the proof divided into two cases:

Case(1): If $v = v_1$ we have always the vertex that represented to the end point of the form v_{m+1} that adjacent to v_1 not dominating with any vertex in S' (i.e. $N[S'] \neq V$).

So clearly that S' is not dominating set.

Case(2): If v any vertex such that

$v = \left\{v_{1+3k} \middle| k = 1,2,3,\dots,\left\lfloor\frac{m+2}{3}\right\rfloor - 1\right\}$ that mean $v \neq v_1$; also we have always minimum one vertex of the form $v_{(1+3k)-1}$ not dominating with any vertex in S' (i.e. $N[S'] \neq V$).

So S' is not dominating set.

Hence S is minimum

So $\gamma(T_{m,1}) = |S| = \left\lfloor\frac{m+2}{3}\right\rfloor$. $\square$

In the next theorem we except the case that we discuss it in proposition (3) for $m \neq 4$ and $n \neq 2$ (i.e. $\gamma(T_{4,2}) = 2$).

Theorem (3): For $m \geq 3$, $n \geq 2$ and (for $m \neq 4$ and $\neq 2$) the domination number of Tadpole graph $T_{m,n}$ is:

$$\left\lfloor\frac{m+2}{3}\right\rfloor + \left\lfloor\frac{n+1}{3}\right\rfloor \text{(i.e. } \gamma(T_{m,n}) = \left\lfloor\frac{m+2}{3}\right\rfloor + \left\lfloor\frac{n+1}{3}\right\rfloor\text{).}$$

Proof: We indicate the vertices of this graph as two sets the first refer to the vertices of the cycle graph C_m as $\{v_1, v_2, v_3, \dots, v_m\}$ and the second to the vertices of path graph P_n as $\{u_1, u_2, u_3, \dots, u_n\}$.

So the vertices of this graph indicated as:

$V(T_{m,1}) = \{v_1, v_2, v_3, \dots, v_m\} \cup \{u_1, u_2, u_3, \dots, u_n\}$. (See Fig.7)

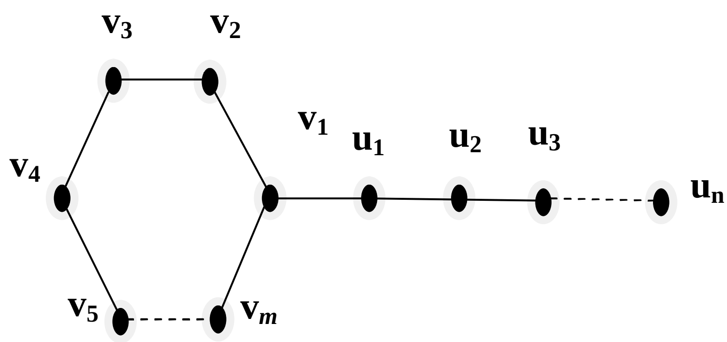

Fig.7: Tadpole graph $T_{m,n}$

we given the dominating set of this graph by a set S:

$S = S_1 \cup S_2$, where

$S_1 = \left\{ v_{1+3k} | k = 0,1,2,3,\ldots, \left\lfloor \frac{m+2}{3} \right\rfloor - 1 \right\}$

S_2 has two cases dependent upon n: If n any number within the sequence

$n = \{2,5,8,11,\ldots,2+3(i-1) \,|\, i = 1,2,3,4,\ldots\}$

We have $S_2 = \left\{ u_{3j} | j = 1,2,3,\ldots, \left\lfloor \frac{n+1}{3} \right\rfloor - 1 \right\} \cup \{u_n\}$

Otherwise $S_2 = \left\{ u_{3j} | j = 1,2,3,\ldots, \left\lfloor \frac{n+1}{3} \right\rfloor \right\}$

The minimality of the set S also follows from theorem (1) similarly in theorem (2).

For any $v \in S$ that is v either inS_1 or v in S_2 so the proof divided into two cases:

Case (1): If $v \epsilon S_1$ (that v is any vertex in C_m)

the proof of this case is the same in theorem(2)

So clearly that S' is not dominating set (by the definition of dominating set) (i.e. $[S'] \neq V$).

Case (2): $v \epsilon S_2$ (that v is any vertex in P_n)

We have always minimum one vertex of the form $u_{(3j)-1}$ not dominating with any vertex in S'

So S' is not dominating set.

So chosen S in this way ensures that there is no proper subset of S dominating $T_{m,n}$

$\therefore S$ minimum

$\gamma(T_{m,n}) = |S| = \left\lfloor \frac{m+2}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor \square$

On domination numbers of Book graph B_m and Stacked Book $B_{m,n}$ graph:

Lemma (1): For $m \geq 3$, the domination number of Book graph B_m is 2

(i.e. $\gamma(B_m) = \gamma(S_{m+1} \times P_2) = 2$).

Proof: Book graph B_m is the Cartesian product $B_m = S_{m+1} \times P_2$ consist always of two $(m+1)$-stars union $(m+1)$ path P_2 having $2(m+1)$ vertices so let the vertices of this graph indicated as: $V(B_m) = \{1,2,\dots,m,m+1,m+2,m+3,\dots 2m+1,2m+2\}$.Note that $m+1$ and $2(m+1)$ refer to the center of these two stars

(See Fig.8).

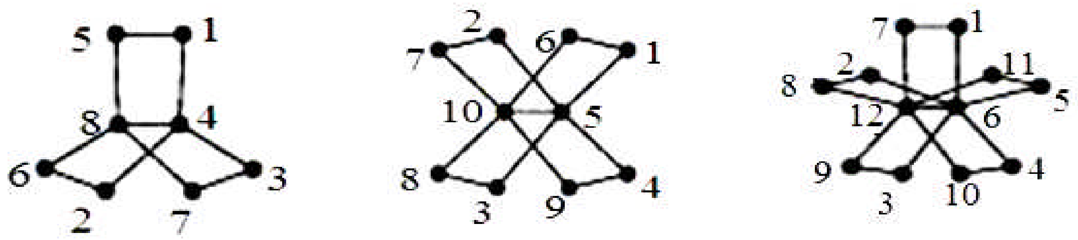

Fig.8

We have given the dominating set of this graph as follows:

$S = \{(m+1), 2(m+1)\}$

It is very easy to proof the minimality of this set.

Suppose S is not minimum this implies that there is a proper sub set dominating B_m .

If $v \in S$ that is either $v = m+1$ or $v = 2(m+1)$

If $v = m+1$, let $S' = S - v$ we have all the vertices that adjacent to v not dominating with the vertex $2(m+1) = S - v = S'$

So S' is not dominating set.

Similarly, for $v = 2(m+1)$

So chosen S in this way ensures that there is no proper subset of S dominating B_m

∴ S minimum

$\gamma(B_m) = |S| = 2$. □

Theorem (4): the domination number of Stacked Book graph $B_{m,n}$ is n

(i.e. $\gamma(B_{m,n}) = \gamma(S_{m+1} \times P_n) = n$).

Proof: For $m \geq 3$ and $n \geq 2$ the proof is similar to that given in Lemma (1) by taking the dominating set of this graph the centers of all stars

$S = \{m+1, 2(m+1), 3(m+1), \dots, n(m+1)\}$. □

Determination and testing the domination numbers of Tadpole graph $T_{m,n}$, Book graph B_m and Stacked Book $B_{m,n}$ graph using matlab:

In this section we write programs in matlab for the purpose of making sure and certainty of the validity of our theoretical results to compute the dominating sets S and the domination number of Tadpole graph, Book graph and Stacked Book graphs. Moreover draw these results in the same graph by referring to the vertices that domination this graphs with different color from the other vertices.
We compute examples for special cases of Tadpole graph, Book graph and Stacked Book graphs by generalizing the results for any (n) and (m) where in every execution of this programs request an input n or m for this graphs.

For the domination numbers of Tadpole graph $T_{m,n}$:

By the program below we demonstrate how to find the domination number of Tadpole graph $T_{m,n}$ using matlab:

```
m=input('m=');
n=input('n=');
a=diag(ones(m-1,1),1);
b=diag(ones(m-1,1),-1);
v=a+b;
v(m,1)=1;
v(1,m)=1;
f=zeros(m,n);
f(1,1)=1;
l=diag(ones(n-1,1),1);
j=diag(ones(n-1,1),-1);
w=l+j;
d=[[v] [f];[f]' [w]];
g=graph
set_matrix(g,d)
distxy(g)
[d,s]=dom(g)
distxy(g)
disp(['the domination number is=',int2str(d)])
distxy(g)
nots=setdiff(1:nv(g),s);
p=partition({s,nots});
distxy(g)
clf;distxy(g)
cdraw(g,p,'-','crgb')
```

Fig.(9) explain the particular results of the domination number of Tadpole graph by using matlab we obtain the same theoretical results.

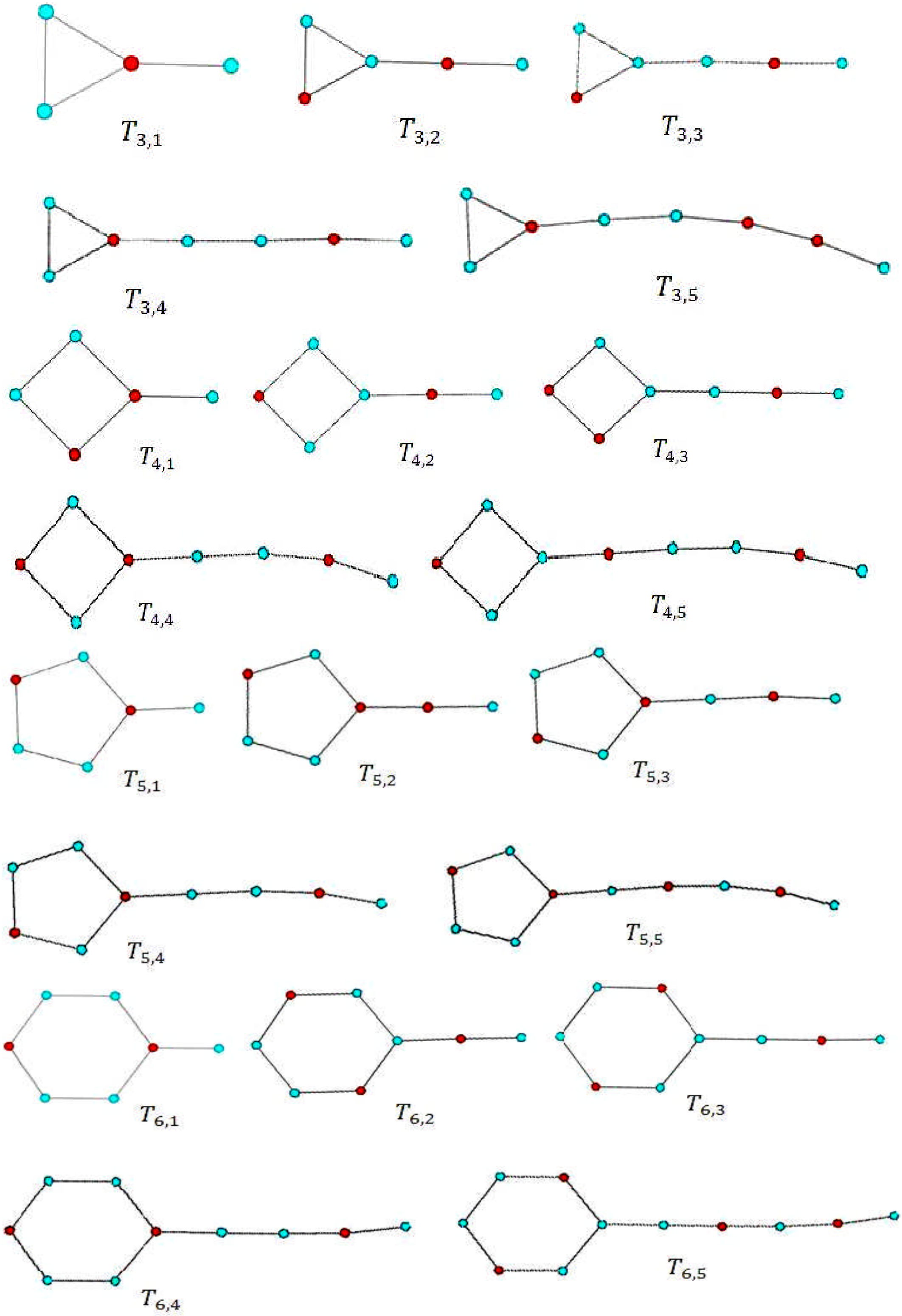

Fig.(9)

For the domination numbers of Book graph B_m and Stacked Book $B_{m,n}$:

By using matlab the program below demonstrate how to find the domination numbers of Stacked Book graph $B_{m,n}$ and by input $n=2$ for any m we obtain the the domination numbers of Book graph B_m :

```
m=input('m=');
n=input('n=');
a=diag(ones(m-1,1),1);
b=diag(ones(m-1,1),-1);
v=a+b;
f=ones(1,m);
c=[0 [f];[f]' [v]];
g=graph
set_matrix(g,c)
h=graph
path(h,n)
k=graph
cartesian(k,g,h)
distxy(k)
[d,s]=dom(k)
disp(['the domination number is=',int2str(d)])
nots=setdiff(1:nv(k),s);
p=partition({s,nots});
distxy(k)
clf;cdraw(k,p,'-','rcrgb')
```

Fig.(10) explain the particular results of the domination number of Book graph B_m then Stacked Book graph $B_{m,n}$ which is the same theoretical result.

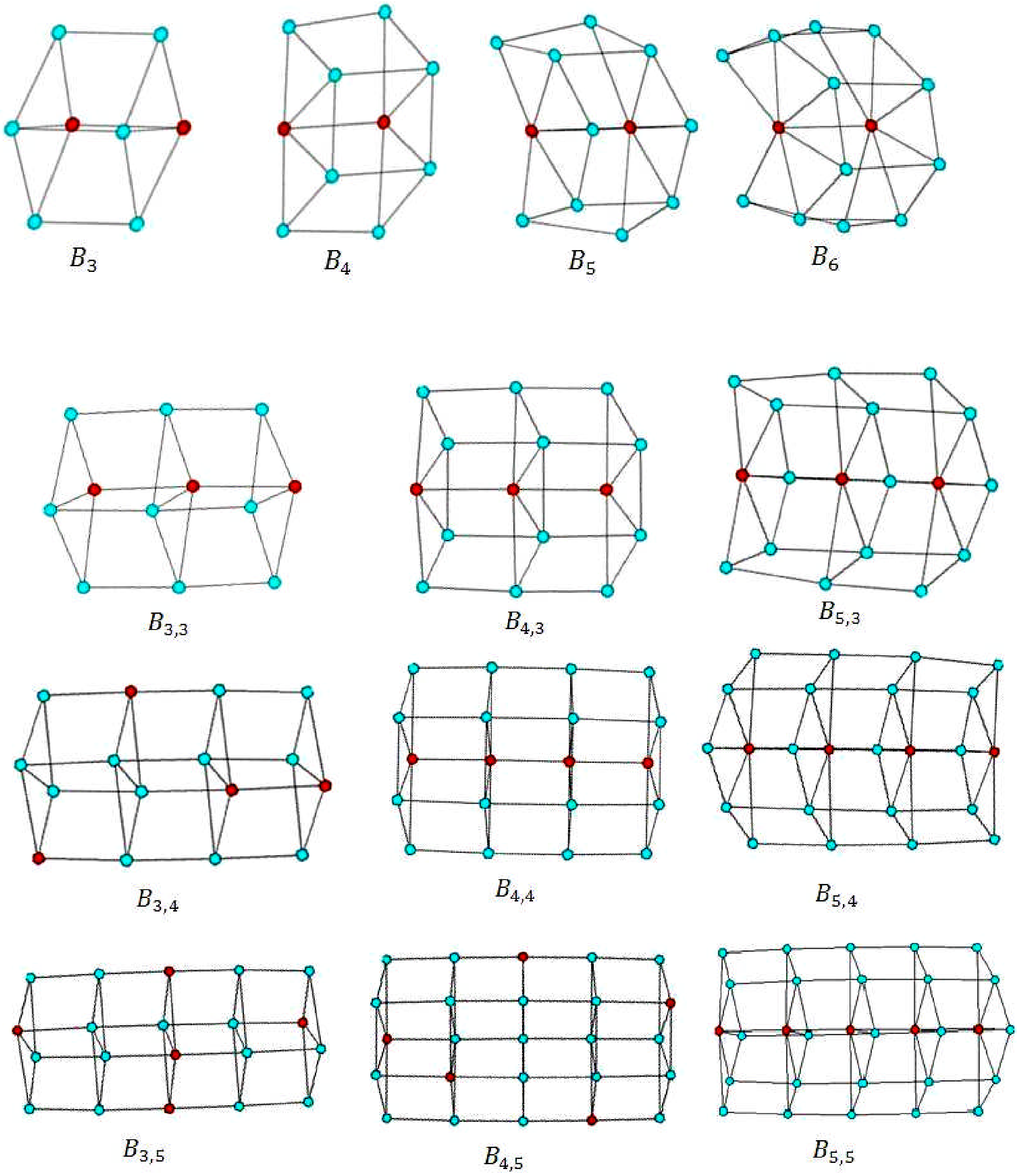

Fig.(10)

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